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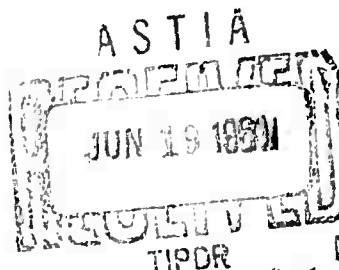
THE MICROWAVE DIAGNOSIS OF A COLUMN OF IONIZED GAS

J. R. Barthel

May 1961

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THE MICROWAVE DIAGNOSIS OF A COLUMN OF IONIZED GAS

J. R. Barthel

ABSTRACT

The inverse scattering problem of deducing the structure of a column of ionized gas from its microwave scattering properties is solved for a homogeneous column and a column which is sufficiently dilute that the electrons may be assumed to be independent scatterers. Criteria are presented which may be applied to experimental data to determine which of the above models (if either) is applicable.

INTRODUCTION

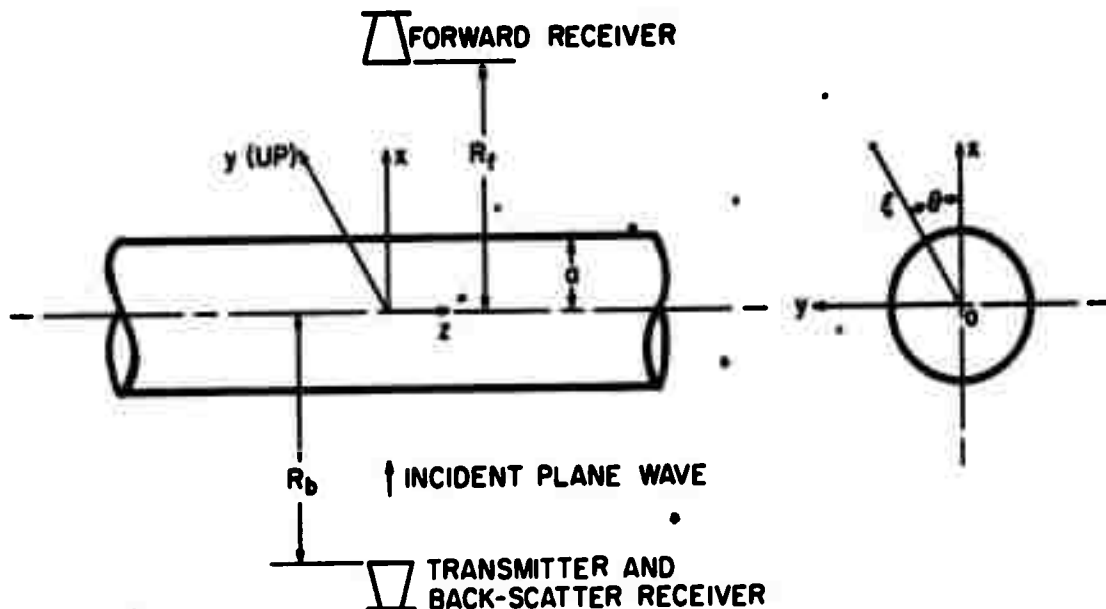
A summary and an extension of methods for determining the signals scattered forward and backward by a column of ionized gas, with known properties, immersed in a known incident plane microwave field is presented. A method is developed for solving the inverse problem of determining the properties of the gas column from measuring the forward and backward scattered signals for negligible absorption. The direct and inverse problems are considered for two models:

1. The column of ionized gas is treated as a homogeneous dielectric constant is an arbitrary complex number;
2. The column is treated as a large number of individual electrons which scatter the incident field independently. These electrons are distributed with a number density which may be a strong function of the distance from the center of the column but a weak function of the distance along the axis.

The theory is applicable to the interpretation of measurements made of the signal scattered forward and backward by the ionized wake of a hypervelocity pellet.

SCATTERING BY DIELECTRIC CYLINDERS

Consider a dielectric cylinder in a vacuum. The incident plane wave propagates in a direction normal to the axis of the cylinder as shown below.



An approach to this problem is to expand the incident plane wave as an infinite series of cylindrical waves.¹ The reflected and refracted waves are similarly represented, but the coefficients in these two series remain undetermined. By applying the boundary conditions at the surface of the cylinder, it is possible to find these coefficients. However, for arbitrary variations of the dielectric constant $K(\xi)$, an equation of the following type² must be solved for each integer n in the series from $-\infty$ to ∞

$$(\xi f_n')' + (k^2 K(\xi) \xi - n^2/\xi) f_n = 0.$$

- Each choice of the function $K(\xi)$ presents us with a very formidable computing task, even if the series converges rapidly. Thus, in practice, the study of the dielectric cylinder is made only for a very good conductor (refracted

fields vanish) and a uniform dielectric [$K(\xi) = \text{constant}$]; the former will be represented here as a special case of the latter by taking the limit $|K| \rightarrow \infty$.

We will consider only the transverse magnetic (TM) case, in which the incident wave is polarized in the Z-direction. The transverse electric (TE) wave, polarized perpendicular to the Z-axis, has been treated by Panofsky and Phillips¹ and by Eshleman² as well as by others for some of the special cases which we will discuss.

The general formula for the electric field at all points outside a uniform dielectric cylinder immersed in a uniform, infinite plane (TM) wave is the sum of the incident and the scattered fields:

$$E = E_z = E_z^i + E_z^s = e^{-i\omega t} |E_z^i| \left\{ \left[e^{ik\xi \cos \theta} - \sum_{n=-\infty}^{\infty} i^n H_n^{(1)}(k\xi) e^{in\theta} F_n(ka; K) \right] \right\} \quad (1)$$

where

$$F_n(ka; K) = \frac{J_n(\sqrt{K}ka) J_n'(ka) - \sqrt{K} J_n'(\sqrt{K}ka) J_n(ka)}{J_n(\sqrt{K}ka) H_n^{(1)'}(ka) - \sqrt{K} J_n'(\sqrt{K}ka) H_n^{(1)}(ka)},$$

J_n , $H_n^{(1)}$ are the Bessel and Hankel functions of the first kind, order n , and $k = \omega/C$. In all cases which interest us, the point of observation is far enough from the cylinder that $H_n^{(1)}(k\xi)$ may be represented by the asymptotic expression

$$H_n^{(1)}(k\xi) = (-1)^n \sqrt{\frac{2}{\pi k\xi}} \exp \left[i \left(k\xi - \frac{\pi}{4} \right) \right] \quad (2)$$

which holds $k\xi \gg 1$. Here, Eq. (1) becomes

$$E = |E_z^i| e^{-i\omega t} \left\{ e^{ik\xi \cos \theta} - \sqrt{\frac{2}{\pi k\xi}} e^{i(k\xi - \pi/4)} \sum_{n=-\infty}^{\infty} e^{in\theta} F_n(ka; K) \right\} \quad (3)$$

It is convenient to define a backscattering reflection coefficient ρ_b and a forward scattering coefficient ρ_f such that

$$\left| E_z^s \right|_{\theta=\pi} = \left| E_z^i \right| \sqrt{\frac{2}{\pi k R_b}} \rho_b ; \left| E_z^s \right|_{\theta=0} = \left| E_z^i \right| \sqrt{\frac{2}{\pi k R_f}} \rho_f \quad (4)$$

From Eqs. (3) and (4), we find

$$\rho_b = \sum_{n=-\infty}^{\infty} (-1)^n \left[F_n(ka; K) \right] \quad (5)$$

and

$$\rho_f = - \sum_{n=-\infty}^{\infty} \left[F_n(ka; K) \right] \quad (6)$$

In general, the evaluation of expressions (5) and (6) is very tedious and must be repeated for each given set of values of ka and K ; therefore, it will be useful to examine certain limiting cases.

Limiting Values of K

The dielectric constant K may take a variety of forms. For the usual non-conducting dielectric medium, it is a real number $K_r > 1$; for a conductor,

$K = 1 + \frac{1}{\epsilon \omega} \frac{\sigma}{\omega}$; for an ionized gas, $K = 1 - \frac{\omega_p^2/\omega^2}{1 - i\nu/\omega}$ where ω_p is the plasma

frequency ($= \frac{N(\text{electrons}/m^3)e^2}{m_e \epsilon c^2}$) and ν/ω represents the attenuation by

collision and by radiation.

In a very good dielectric or a very good conductor, we find $|K| \gg 1$. Then Eqs. (5) and (6) approach the following form, as $|K| \rightarrow \infty$,

$$\rho_b = \sum_{n=-\infty}^{\infty} (-1)^n \frac{J_n(ka)}{H_n^{(1)}(ka)} \quad (7)$$

$$\rho_f = - \sum_{n=-\infty}^{\infty} \frac{J_n(ka)}{H_n^{(1)}(ka)} \quad (8)$$

(Note: Eq. (7) is shown graphically by Eshleman,² pp. 111 and 112.)

In an ionized gas near the critical frequency, $|K| \ll 1$. As $|K| \rightarrow 0$, however, no terms in Eq. (1) may be dropped. It may be seen by expanding $J_n(\sqrt{K} ka)$ and $J_n'(\sqrt{K} ka)$ for small arguments that the two terms in the numerator of Eq. (1) and the two terms in the denominator are all of the same order in K .

Limiting Values of ka

The number $ka = \omega a/c$ is a real number. We consider two limits:

1. $ka \ll 1$, i.e., the long-wave limit;
2. $ka \gg 1$, the optical limit.

For small values of ka , we find the following results? If $|K|$ is not too large, so that $|\sqrt{K} ka| \ll 1$ as well as $ka \ll 1$, then

$$\rho_b = \rho_f = \frac{i\pi (K - 1)(ka)^2}{4} \quad (9)$$

If $|K| \gg 1$ then

$$\rho_b = \rho_f = \frac{i\pi}{2 \log \left(\frac{\gamma ka}{2} \right)} \quad (\log \gamma \approx 0.5772; \gamma \approx 1.781). \quad (10)$$

If $|K| \ll 1$, then

$$\rho_b = \rho_f = -\frac{i\pi}{4} (ka)^2 \quad (11)$$

Note that Eq. (10) is valid only if $|K|$ is so large that $|K| (ka)^2 \log (ka) \gg 1$.

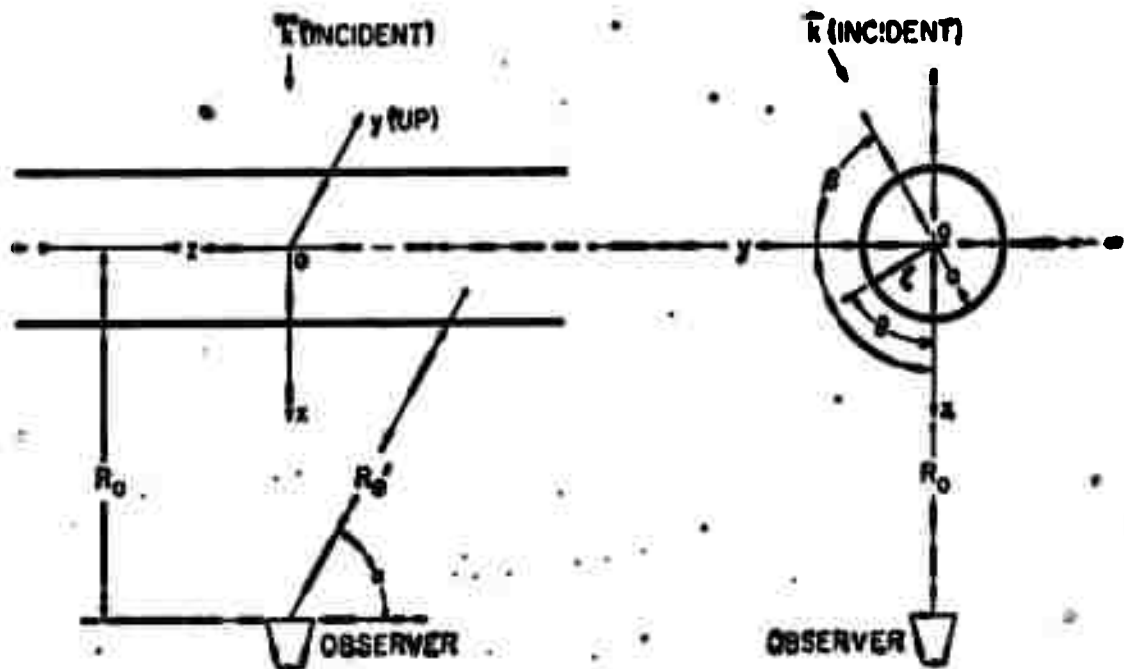
For large values of ka , it is difficult to approximate the infinite series of Eq. (1). However, the laws of geometrical optics hold in this limit. The reflection coefficient from a perfectly reflecting cylinder

(for which $|K| \gg 1$) is readily found² to be

$$\rho_0 = \sqrt{\frac{\pi k a}{4}}. \quad (12)$$

ELECTRON SCATTERING THEORY

Consider an ionized column of gas with variable electron density, $N(\xi)$ which is immersed in a plane microwave field. The incident wave propagates in a direction normal to the axis of the column as shown below.



The expression of Eshleman's² Eq. (64) for the scattered electric field at the point of observation for $\beta = 0$ may be extended for arbitrary β to the following form:

$$\mathbf{E}_z^s = -\frac{\mu_0 e^2 |\mathbf{E}^i|}{4\pi m_e} \int_0^a \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{N(\xi) \exp \{ik [R_0^2 - \xi \cos(\beta - \theta)] - i\omega t\} \sin^2 \alpha d\xi d\theta dz}{R_0}, \quad (13)$$

here we have assumed that all electrons are subject to an incident wave of the same strength. This assumption holds only for sufficiently dilute columns in which $v \ll \omega$. Note that E_x and E_y at the observer are zero because of symmetry.

If a back-scattered signal is being observed ($\beta = 0$), and if $R_0 \gg a$, then the following approximate result holds

$$E_z^s \Big|_{\beta=0} = \frac{-i\mu_0 e^2}{4m_e} \cdot 2\pi \left| E_z^i \right| \exp \left[i(kR_0 - \frac{\pi}{4} - \omega t) \right] \sqrt{\frac{2}{\pi k R_0}} \int_0^a N(\xi) J_0(2k\xi) \xi d\xi; \quad (14)$$

or

$$\mu_0 = A \int_0^a N(\xi) J_0(2k\xi) \xi d\xi \quad (15)$$

where

$$A = - \frac{i\mu_0 e^2}{2m_e}$$

If the forward-scattered signal is being observed ($\beta = \pi$), and $R_0 \gg a$, then we find

$$\begin{aligned} E_z^s \Big|_{\beta=\pi} &= A \left| E_z^i \right| \exp \left[i(kR_0 - \frac{\pi}{4} - \omega t) \right] \sqrt{\frac{2}{\pi k R_0}} \int_0^a N(\xi) \xi d\xi \\ &= \frac{A}{2\pi} \left| E_z^i \right| \exp \left[i(kR_0 - \frac{\pi}{4} - \omega t) \right] \sqrt{\frac{2}{\pi k R_0}} q \end{aligned} \quad (16)$$

where q is the number of electrons per unit axial length. The total field observed includes the incident field, and may be expressed as

$$E_z \Big|_{\beta=\pi} = \left| E_z^i \right| \exp \left[i(kR_0 - \omega t) \right] \left[1 + \frac{\mu_0 e^2}{4m_e} \exp \left(-i \frac{3\pi}{4} \right) \sqrt{\frac{2}{\pi k R_0}} q \right] \quad (17)$$

Note that

$$\left| E_z \right|_{\beta=\pi} < \left| E_z^i \right|$$

as expected. However, this effect is not due to absorption, which is neglected but is due to the fact that the net phase of the scattered signal opposes the phase of the incident field when observed in the direction $\beta = \pi$.

Equation (17) for the field at the forward observation point holds for any electron distribution $N(\xi)$. Equation (15) for the reflection coefficient has been evaluated² for several special cases:

$$\text{Narrow Column, } ka \ll 1 : \rho_b = \frac{A}{2\pi} q \quad (18)$$

$$\text{Homogeneous Cylinder, } N(\xi) = N_0 : \rho_b = \frac{A}{2\pi} q \frac{J_1(2ka)}{ka} \quad (19)$$

$$\text{Gaussian Distribution, } N(\xi) = N_0 \exp[-\xi^2/a^2] :$$

$$\rho_b = \frac{A}{2\pi} q \exp[-(ka)^2] \quad (20)$$

It has also been found that for an exponential distribution, $N(\xi) = N_0 e^{-\xi/a}$

$$\rho_b = A N_0 a^2 \int_0^\infty e^{-\xi/a} J_0(2ka \frac{\xi}{a}) \frac{\xi}{a} \frac{d\xi}{a} = \frac{A}{2\pi} \frac{q}{\sqrt{1 + 4k^2 a^2}} \quad (21)$$

For arbitrary functions $N(\xi)$, Eq. (15) may be evaluated approximately by expanding $J_0(2k\xi)$ in series

$$J_0(2k\xi) = 1 - \frac{(k\xi)^2}{(1!)^2} + \frac{(k\xi)^4}{(2!)^2} - \frac{(k\xi)^6}{(3!)^2} + \dots \quad (22)$$

$$\frac{\rho_b}{A} = \int_0^a N(\xi) \xi d\xi - k^2 \int_0^a N(\xi) \xi^3 d\xi + \frac{k^4}{4} \int_0^a N(\xi) \xi^5 d\xi - \frac{k^6}{36} \int_0^a N(\xi) \xi^7 d\xi + \dots$$

This series converges rapidly if ka is not large.

If at some part of the column the electron density is large enough that $K_{\text{real}} < 0$, then the cylinder of radius ξ_c reflects almost perfectly and may be treated as a metal cylinder, if $K_r \leq 0$ for $\xi \leq \xi_c + \delta$. Here δ is the

skin depth, i.e., the depth inside the surface $K_r = 0$ at which the electric field attenuates by a factor e^{-1} . The region $\xi > \xi_c$ may still be treated as a dilute medium, but the above analysis is not strictly applicable because the field reflected from ξ_c , as well as the incident field, acts on the individual electron scatters.

THE INVERSE PROBLEM

The problem of determining the structure of the column from the scattered signals is more difficult. We wish to obtain the radius a and the dielectric constant K , which may be a function of ξ ; alternatively, as the problem is posed in the previous section, we wish to determine $N(\xi)$.

Only certain simple configurations may be analyzed successfully. The first step is to determine which, if any, of the following simple models is most consistent with the data: the dilute column of scattering electrons, the narrow uniform dielectric cylinder, or the perfectly reflecting cylinder. Our choice of a model must be guided by a priori knowledge of the column, as obtained from approximate calculations for the known conditions; this choice must be substantiated by making a comparison with the experimental results.

If estimates indicate that collisions are significant or that q exceeds a certain limit (which will be obtained later) then we may postulate the narrow, uniform, dielectric cylinder model and compare it with experiment. From Eq. (9) we obtain

$$\rho_b = \rho_f = \frac{1\pi a^2 \omega_p^2}{c^2(1 - iv/\omega)} \quad (23)$$

We may determine $(a \omega_p)$ and v by applying Eq. (23) at two frequencies.* The model becomes convincing only if there are a number of measuring frequencies, so that the result may be checked several different ways. Note, however, that we may apply Eq. (23) only at frequencies for which ka is somewhat less than unity. The indeterminacy may be resolved if a frequency can be found which is low enough that the skin depth δ is much less than

*Note that $(a \omega_p)^2 = q \times \text{known constant}$, i.e., we then know the number of electrons per unit axial length.

the column radius; we find $\delta_{\min} \approx c/\omega_p$ (as $\omega \rightarrow 0$). If $c/\omega_p \ll a$, we may use Eq. (7) to find a , since the column is then effectively a perfectly reflecting cylinder; if not, the indeterminacy will remain unless some independent way of finding a or ω_p is found.

If the column is not sufficiently uniform for the above method to succeed, then it appears that a simple microwave analysis of the column structure is not feasible unless the dilute column model (independent scattering electrons) may be applied. However, this model applies only when absorption [by collisions with neutral particles or by electron resonance ($\omega \leq \omega_p$)] is small. For this model Eq. (17) leads to the following relation for the forward-scattered power ratio*

$$\frac{P_f}{P_{f0}} = 1 - \frac{1}{2} \left(\frac{\mu_0 e^2}{m_e} \right) \sqrt{\frac{1}{\pi k R_f}} q + \frac{1}{8} \left(\frac{\mu_0 e^2}{m_e} \right) \left(\frac{1}{\pi k R_f} \right) q^2 \quad (24)$$

where P_{f0} refers to the absence of the ionized column. The solutions of Eq. (24) for q , from the measured values of P_f/P_{f0} , should give nearly the same answer for each $k > \omega_p(\max)/c$, if we are to be justified in using the dilute column model.

If application of Eq. (24) for various k justifies the dilute column model, we may then proceed to determine $N(\xi)$ by solving Equation (15)

$$\rho_b(k) = A \int_0^\infty N(\xi) J_0(2k\xi) \xi d\xi. \quad (25)$$

The solution is found by applying the Fourier-Bessel integral³ which states

$$f(z) = \int_0^\infty J_m(kz) k dk \int_0^\infty f(\xi) J_m(k\xi) \xi d\xi \quad (m > -\frac{1}{2}) \quad (26)$$

* Note that if q exceeds a certain value, which need not correspond to $\omega_p > \omega$, there is a focusing effect, i.e., $P_f/P_{f0} > 1$.

After applying Eq. (26), we find

$$\begin{aligned}
 N(\xi) &= \int_0^{\infty} J_0(2k\xi) (2k) 2dk \int_0^{\infty} N(x) J_0(2kx) xdx \\
 &= \frac{4}{A} \int_0^{\infty} J_0(2k\xi) \rho_b(k) kdk.
 \end{aligned} \tag{27}$$

There is a question of the validity of the application of Eq. (27). There is a frequency ω' , or a wave number k' , below which the integrand does not have the form which corresponds to a dilute column (ω' is the plasma frequency corresponding to N_{\max}). If the region $k < k'$ does not make the dominant contribution to the integral, we may resolve this question by employing the following scheme. First we determine a trial function $N(\xi)$ by Eq. (27) by using the experimental values of $\rho_b(k)$. Then we consider the value N_{\max} , which corresponds to

$$\omega'^2 = \omega_p^2(\max) = \frac{e^2 N_{\max}}{m_e \epsilon} = (k'c)^2. \tag{28}$$

We assume $N_{\max} = N(0)$, for simplicity in developing the scheme, although $N_{\max} \neq N(0)$ is treated by a straightforward extension of the following. Now notice that

$$N(0) = \frac{4}{A} \int_0^{\infty} \rho_b(k) kdk = N_{\max} \tag{29}$$

We wish to adjust $\rho_b(k)$, for $k \leq k'$, to the form it would have if the column continued to act as a number of independent scattering electrons, i.e., we wish to remove the effect of absorption and reflection from regions in which $\omega_p > \omega$. We know N_{\max} , we find k' , and then we analytically extend the function $\rho_b(k)$ (for $k \geq k'$) to the interval $0 \leq k \leq k'$. In so doing, we may utilize the fact that for the dilute column model with $ka \ll 1$, $\rho_b = qA/2\pi$, where q is estimated by using the trial function $N(\xi)$ in the expression

$$q = 2\pi \int_0^{\infty} N(\xi) \xi d\xi.$$

By using the adjusted function $\rho_b(k)$, we then repeat the above procedure, and again find a trial function $N(\xi)$, find k' , and adjust $\rho_b(k)$ for $k \leq k'$. We may continue to repeat the procedure until convergence is evident. However, it is doubtful that the scheme will lead to accurate values of $N(\xi)$ unless the function to which $N(\xi)$ converges is not greatly different from the result of the first correction; i.e., we must restrict k' to be small enough so that the major contribution to the integral for $N(\xi)$ at each ξ is in the region $k > k'$.

We may estimate the severity of the above restriction by considering the case $k'a \ll 1$; then $\rho_b(k \leq k') = qA/2\pi$ [Eq. (18)]. The above restriction requires that

$$N(0) \gg \frac{4}{A} \int_0^{k'} \rho_b k dk = \frac{2q}{\pi} \frac{(k')^2}{2} \frac{qe^2 N(0)}{\pi m_e c^2} \quad (30)$$

or $q \ll 10^{14}$ electrons/meter, where we have employed Eqs. (28) and (29). Therefore, we may expect successful application of Eq. (27) to the inverse problem only if approximate calculations for the prevailing conditions in the column indicate that q is considerably less than 10^{14} electrons/meter of column length. This restriction may be unnecessarily severe, because the value of ρ_b assumed for $k < k'$ is the maximum that any of the models in Eqs. (18) to (21) may attain. Nevertheless, if we assume, for example, a Gaussian distribution, the above restriction leads to

$$N(0) \gg \frac{4}{A} \int_0^{k'} \rho_b k dk = \frac{4q}{2\pi} \int_0^{k'} e^{-(ka)^2} k dk = N(0) [1 - e^{-(k'a)^2}] \quad (31)$$

$$\text{or } (k'a)^2 \ll 1$$

which will lead to the same restriction as Eq. (30).

There is less restriction on the validity of the scattering electron theory for the direct problem. If the plasma frequency ω_p is less than ω everywhere in the column, and if $v/\omega \ll 1$, the theory is valid for the direct problem no matter how large q may be. Eshleman² has shown that for a column with $\omega_p > \omega$ and $kr_0 < 1$, where r_0 is defined by $q = \pi N_{\max} r_0^2$,

then it is required that $q \ll 10^{14}$ electrons/meter to validate the theory; if $(kr_0) > 1$ and $\omega_p > \omega$, it is required that $q \ll (kr_0)^2 10^{14}$ electrons/meter. The more stringent requirements for validity of the theory in the inverse problem are caused by the necessary consideration of very small as well as very large values of ω .

SUMMARY AND CONCLUSIONS

We have first summarized results derived elsewhere for microwave scattering from columns which are either

1. uniform dielectrics (K may be complex), or
2. sufficiently dilute that the electrons scatter independently.

These results were then extended, for the second case, to forward-scattering. It was found that the ratio of the power of the forward-scattered signal to its value in the absence of a scattering column may exceed unity, i.e., there is a focusing effect; it was also found that, neglecting collisions, this power ratio depends only on the number of electrons per unit length of the column.

The inverse problem of determining the column structure from its scattering properties was also studied, for both cases. In Case (1.) we may determine at the least, the collision frequency ν of the electrons and the number of electrons q per unit axial length; if the column reflects perfectly at sufficiently low frequencies, we may measure the column radius a , and hence deduce the electron density N from the value of q . In Case (2.) it was shown that the electron density $N(\xi)$, which may vary radially, may be determined from the reflection coefficient ρ_b by applying the Fourier-Bessel integral, provided q is considerably less than 10^{14} electrons/meter.

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